

# DYNAMICS OF CONVERGENT POWER SERIES ON THE INTEGRAL RING OF A FINITE EXTENSION OF $\mathbb{Q}_p$

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**ABSTRACT.** Let  $K$  be a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers and  $\mathcal{O}_K$  be its integral ring. The convergent power series with coefficients in  $\mathcal{O}_K$  are studied as dynamical systems on  $\mathcal{O}_K$ . A minimal decomposition theorem for such a dynamical system is obtained. It is proved that there are uncountably many minimal subsystems, provided that there is a minimal set consisting of infinitely many points. In particular, the complete detailed minimal decompositions of all affine systems are derived.

## 1. INTRODUCTION

This paper contributes to the theory of  $p$ -adic dynamical systems which has recently been intensively developed. We refer to Silverman's book [22] and Anashin and Khrennikov's book [4] for this development.

Let  $p$  be a prime number. Let  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers and  $|\cdot|_p$  be the  $p$ -adic absolute value on  $\mathbb{Q}_p$ . Consider a finite extension  $K$  of  $\mathbb{Q}_p$  with  $d = [K : \mathbb{Q}_p]$  being the degree of the extension. The extended absolute value of  $K$  is still denoted by  $|\cdot|_p$ . Let  $\mathcal{O}_K := \{x \in K : |x|_p \leq 1\}$  be the local ring of  $K$ . Define by

$$\mathcal{O}_K\langle x \rangle := \left\{ \sum_{i \geq 0} a_i x^i \in \mathcal{O}_K[[x]] : \lim_{i \rightarrow \infty} a_i = 0 \right\}$$

the class of convergent power series with coefficients in  $\mathcal{O}_K$ . Remark that when  $K = \mathbb{Q}_p$ , the class  $\mathcal{O}_K\langle x \rangle$  was called the class  $\mathcal{C}$  by Anashin in [2]. A power series  $\phi \in \mathcal{O}_K\langle x \rangle$  is considered as a topological dynamical system on  $\mathcal{O}_K$ , denoted as  $(\mathcal{O}_K, \phi)$ .

Let  $X$  be a compact metric space and  $T$  be a continuous map from  $X$  to itself. Denote by  $T^{\circ n}$  the  $n$ th iterate of  $T$ , i.e.,

$$T^{\circ n} = \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}.$$

For  $x \in X$ , the forward orbit of  $x$  is the set  $\{T^{\circ n}(x) : n \geq 0\}$ . The system  $(X, T)$  is said to be *minimal* if each orbit is dense in  $X$ .

In the literature, the minimality, or equivalently the ergodicity with respect to the Haar measure, of polynomial dynamical systems on the ring  $\mathbb{Z}_p$  of  $p$ -adic integers (which is the local ring of  $\mathbb{Q}_p$ ) are extensively studied ([1, 2, 3, 5, 7, 10, 11, 12, 14, 15, 18]). See also the monographs [4, 16] and the bibliographies therein. In [12], Fan and Liao proved the following result which asserts that a polynomial with coefficients in  $\mathbb{Z}_p$  and with degree at least 2, as a dynamical system on  $\mathbb{Z}_p$ , admits at most countably many minimal subsystems.

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**Theorem A.** *Let  $\phi$  be a polynomial with coefficients in  $\mathbb{Z}_p$  and with degree at least 2. Then we can decompose the space  $\mathbb{Z}_p$  as*

$$\mathbb{Z}_p = A \sqcup B \sqcup C$$

where  $A$  is the finite set consisting of all periodic points of  $\phi$ ,  $B = \bigsqcup_i B_i$  is the union of all (at most countably many) clopen invariant sets such that each subsystem  $\phi : B_i \rightarrow B_i$  is minimal, and each point in  $C$  lies in the attracting basin of a periodic orbit or of a minimal subsystem.

As for the affine polynomials, Fan, Li, Yao and Zhou [11] showed that apart from some special cases like  $ax$  with  $a^n = 1$ , the systems have similar dynamical structures.

The decomposition in Theorem A is referred to as a minimal decomposition. The topological dynamical structure of a polynomial is hence totally described by its minimal decomposition. There are few works on the minimal decomposition. Multiplications  $f(x) = ax$ , with  $|a|_p = 1$  on  $\mathbb{Z}_p$  ( $p \geq 3$ ) were studied by Coelho and Parry [8]. All the minimal components of a general affine polynomial on  $\mathbb{Z}_p$  were exhibited by Fan, Li, Yao and Zhou [11]. The minimal decomposition for the quadratic polynomials in the case  $p = 2$  was obtained by Fan and Liao [12]. The minimal decomposition for homographic maps on the projective line over  $\mathbb{Q}_p$  was studied by Fan, Fan, Liao and Wang [13].

In the present paper, we would like to study the dynamical systems  $(\mathcal{O}_K, \phi)$  with  $\phi \in \mathcal{O}_K \langle x \rangle$  being a convergent series. As we will show, the results in the finite extension cases are different. If  $K$  is a finite extension of  $\mathbb{Q}_p$  and  $K \neq \mathbb{Q}_p$ , then  $(\mathcal{O}_K, \phi)$  are always non-minimal. Moreover, there are uncountably many minimal subsystems of  $(\mathcal{O}_K, \phi)$  for each  $\phi \in \mathcal{O}_K \langle x \rangle$ , if there is a non-trivial minimal subset (i.e. a minimal subset consisting of infinitely many points).

Let  $e$  be the ramification index of  $K$  over  $\mathbb{Q}_p$ . We will prove the following theorem. The reader is referred to Section 4 for the definition of type  $(k, e)$  of a subsystem. Actually, a type  $(k, e)$  subsystem contains uncountably many minimal subsystems which have the same dynamical structure (in fact, they are all conjugate to the same adding machine, see Section 4).

**Theorem 1.1.** *Let  $\phi \in \mathcal{O}_K \langle x \rangle$ . Suppose that for each  $n \geq 1$ ,  $\phi^{\circ n}$  is not identity. Then we have the following decomposition*

$$\mathcal{O}_K = A \sqcup B \sqcup C$$

where  $A$  is the finite set consisting of all periodic points of  $\phi$ ,  $B = \bigsqcup_i B_i$  is the union of all (finite or countably many) clopen invariant sets such that each  $B_i$  is a finite union of balls and each subsystem  $\phi : B_i \rightarrow B_i$  is of type  $(k\ell p^n, e)$  for some positive integers  $1 \leq k \leq p^f$ ,  $\ell | (p^f - 1)$ , and  $n \in \mathbb{N}$ . Each point in  $C$  lies in the attracting basin of a periodic orbit or of a subsystem of type  $(k\ell p^n, e)$ .

Moreover, there are uncountably many minimal subsystems if  $K \neq \mathbb{Q}_p$ .

To end this section, we point out that the obtained results could be applied to the study of rational maps on  $\mathbb{Q}_p$ . Since a rational map with coefficients in  $\mathbb{Q}_p$  may have fixed points out of  $\mathbb{Q}_p$ , but in a finite extension of  $\mathbb{Q}_p$ . Thus one could study the corresponding dynamics on this finite extension, then translate the results to the restricted subsystem on  $\mathbb{Q}_p$ . A good example is the recent work [13] where multiplication dynamics on a quadratic extension is studied to obtain the dynamical structure of homographic dynamics on  $\mathbb{Q}_p$ .

The paper is organized as follows. In Section 2, some preliminaries on finite extensions of  $\mathbb{Q}_p$  are recalled. In Section 3, we discuss some fundamental properties of the class

$\mathcal{O}_K\langle x \rangle$  of convergent series which will be useful for the proof of the main theorem. The techniques to study the minimality from local to global are fully developed in Section 4. The proof of the main theorem will be given in Section 5. At the end, as an application, we study the affine polynomials in Section 6.

## 2. FINITE EXTENSIONS OF THE FIELD OF $p$ -ADIC NUMBERS

We recall some basic notations and facts of finite extensions of the field of  $p$ -adic numbers. The reader may consult [19, 17, 20, 21] for more information on non-Archimedean fields.

Let  $K$  be a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. Denote by  $d = [K : \mathbb{Q}_p]$  the degree of the extension, i.e., the dimension of  $K$  as a vector space over  $\mathbb{Q}_p$ . The extended absolute value on  $K$  is still denoted by  $|\cdot|_p$ . For  $x \in K^* := K \setminus \{0\}$ , let  $v_p(x) := -\log_p(|x|_p)$  define the valuation of  $x$ , with convention  $v_p(0) := \infty$ . One can show that there exists a unique positive integer  $e$  which is called the *ramification index* of  $K$  over  $\mathbb{Q}_p$ , such that

$$v_p(K^*) = \frac{1}{e}\mathbb{Z}.$$

The extension  $K$  over  $\mathbb{Q}_p$  is said to be *unramified* if  $e = 1$ , *ramified* if  $e > 1$  and *totally ramified* if  $e = d$ . An element  $\pi \in K$  is called a *uniformizer* if  $v_p(\pi) = 1/e$ . The local ring  $\mathcal{O}_K$  of  $K$  is  $\{x \in K : |x|_p \leq 1\}$ , whose elements are called *integers* of  $K$ . The maximal ideal of  $\mathcal{O}_K$  is  $\mathcal{P}_K := \{x \in K : |x|_p < 1\}$ . Denote by  $\mathbb{K}$  the residual class field  $\mathcal{O}_K/\mathcal{P}_K$  of  $K$ . Then  $\mathbb{K} = \mathbb{F}_{p^f}$ , the finite field of  $p^f$  elements where  $f = d/e$ . Let  $C = \{c_0, c_1, \dots, c_{p^f-1}\}$  be a fixed complete set of representatives of the cosets of  $\mathcal{P}_K$  in  $\mathcal{O}_K$ . Then for each uniformizer  $\pi$ , every  $x \in K$  has a unique  $\pi$ -adic expansion of the form

$$x = \sum_{i=i_0}^{\infty} a_i \pi^i \quad (2.1)$$

where  $i_0 \in \mathbb{Z}$  and  $a_i \in C$  for all  $i \geq i_0$ . For convenience of notations, we define  $v_\pi(x) := e \cdot v_p(x)$  for  $x \in K$ . Then  $v_\pi(K^*) = \mathbb{Z}$ .

## 3. POWER SERIES ON $\mathcal{O}_K$

Recall that  $\mathcal{O}_K\langle x \rangle$  is the class of convergent power series with coefficients in  $\mathcal{O}_K$ . In this section, we mainly discuss about the stability of  $\mathcal{O}_K\langle x \rangle$  under the classical operations.

It is easy to check that  $\mathcal{O}_K\langle x \rangle$  is an algebra with respect to addition and multiplication.

**Proposition 3.1.** *The class  $\mathcal{O}_K\langle x \rangle$  is closed with respect to derivation and composition of functions.*

*Proof.* Let

$$\phi(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{O}_K\langle x \rangle.$$

By noticing the facts that  $\lim_{i \rightarrow \infty} i a_i = 0$  and  $\phi'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$ , we have  $\phi' \in \mathcal{O}_K\langle x \rangle$ . Thus  $\mathcal{O}_K\langle x \rangle$  is closed with respect to derivations.

Now we prove that  $\mathcal{O}_K\langle x \rangle$  is closed with respect to compositions of functions. Let  $\phi(x) = \sum_{i=0}^{\infty} a_i x^i, \psi(x) = \sum_{j=0}^{\infty} b_j x^j \in \mathcal{O}_K\langle x \rangle$ . Notice that

$$\phi \circ \psi(x) = a_0 + \sum_{k=1}^{\infty} a_k b_0^k + \sum_{i=1}^{\infty} \left( \sum_{k=1}^{\infty} a_k \sum_{\substack{i_1+i_2+\dots+i_k=i \\ i_1, \dots, i_k \geq 0}} b_{i_1} b_{i_2} \dots b_{i_k} \right) x^i.$$

Since  $\lim_{i \rightarrow \infty} a_i = 0$  and  $\lim_{j \rightarrow \infty} b_j = 0$ , the strong triangle inequality implies

$$\lim_{i \rightarrow \infty} \sum_{k=1}^{\infty} a_k \sum_{\substack{i_1+i_2+\dots+i_k=i \\ i_1, \dots, i_k \geq 0}} b_{i_1} b_{i_2} \dots b_{i_k} = 0.$$

Thus  $\phi \circ \psi \in \mathcal{O}_K\langle x \rangle$ . □

*Remark 1.* Remind that when  $K = \mathbb{Q}_p$ , the class  $\mathcal{O}_K\langle x \rangle$  was called the class  $\mathcal{C}$  in [2] where the stability of derivation for  $\mathcal{C}$  was also proved. However, the stability of composition which was not checked in [2] will be needed in the proof of Theorem 1.1. We point out that the stability of composition is different from the classical stability of composition of convergent series in the field  $K$ . Here, we have to verify the conditions on the coefficients of a convergent series.

For a power series  $\phi(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{O}_K\langle x \rangle$ , the largest integer  $j$  such that  $|a_j|_p = 1$  is called the *Weierstrass degree* of  $\phi$ , denoted by  $\text{wideg}(\phi)$ . If all coefficients of  $\phi$  are in  $\mathcal{P}_K$ , we say that the Weierstrass degree of  $\phi$  is infinite. We will use the following Weierstrass Preparation Theorem (see Sections 5.2.1-5.2.2 of [6]).

**Theorem 3.2** (Weierstrass Preparation Theorem). *Let  $\phi(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathcal{O}_K\langle x \rangle$  be a nonzero convergent series with  $\text{wideg}(\phi) < \infty$ . Let  $j$  be the largest integer such that  $|a_j|_p = \max_{i \geq 0} |a_i|_p$ . Then there is a monic polynomial  $g \in \mathcal{O}_K[x]$  of degree  $j$  and a power series  $h \in \mathcal{O}_K\langle x \rangle$  such that  $\phi = gh$ , and  $h(x) \neq 0$  for all  $x \in \mathcal{O}_K$ .*

#### 4. INDUCED DYNAMICS ON $\mathcal{O}_K/\pi^n \mathcal{O}_K$

This section is devoted to local dynamics on  $\mathcal{O}_K/\pi^n \mathcal{O}_K$  which will deduce the global dynamics on  $\mathcal{O}_K$ .

In the remainder of this paper we require  $\pi$  to be a fixed uniformizer of  $K$  and  $C = \{c_0, c_1, \dots, c_{p^f-1}\}$  be a fixed complete set of representatives of the cosets of  $\mathcal{P}_K$  in  $\mathcal{O}_K$ . Then every  $x \in \mathcal{O}_K$  has a unique  $\pi$ -adic expansion of the form

$$x = \sum_{i=0}^{\infty} a_i \pi^i$$

with  $a_i \in C$  for all  $i \geq 0$ . We remark that the results in this paper do not depend on the choices of  $\pi$  and  $C$ .

Let  $\phi \in \mathcal{O}_K\langle x \rangle$  be a convergent power series of integral coefficients. The dynamics of  $(\mathcal{O}_K, \phi)$  can be derived by those of its induced finite dynamics on  $\mathcal{O}_K/\pi^n \mathcal{O}_K$ . Let  $n \geq 1$  be a positive integer. Denote by  $\phi_n$  the induced mapping of  $\phi$  on  $\mathcal{O}_K/\pi^n \mathcal{O}_K$ , i.e.,

$$\phi_n(x \bmod \pi^n) = \phi(x) \bmod \pi^n.$$

**Theorem 4.1** ([3, 7]). *Let  $\phi \in \mathcal{O}_K \langle x \rangle$  and  $E \subset \mathcal{O}_K$  be a compact  $\phi$ -invariant set. Then  $\phi : E \rightarrow E$  is minimal if and only if  $\phi_n : E/\pi^n \mathcal{O}_K \rightarrow E/\pi^n \mathcal{O}_K$  is minimal for each  $n \geq 1$ .*

By Theorem 4.1, to study the minimality of  $\phi$ , it suffices to study the minimality of all finite dynamics  $\phi_n$ . To this end, we need an induction from the level  $n$  to the level  $n+1$ . In other words, we want to predict the dynamical structure of  $\phi_{n+1}$  at level  $n+1$  from that of  $\phi_n$  at level  $n$ . The idea of predicting comes from the work of desJardins and Zieve [9] where the case  $K = \mathbb{Q}_p$  ( $p \geq 3$ ) was studied. This idea allowed Fan and Liao [12] to give a minimal decomposition theorem (Theorem A in the first section) for any polynomial with coefficients in  $\mathbb{Z}_p$ .

A collection  $\sigma = (x_1, \dots, x_k)$  of  $k$  distinct points in  $\mathcal{O}_K/\pi^n \mathcal{O}_K$  is called a *cycle of  $\phi_n$  of length  $k$*  or a  *$k$ -cycle at level  $n$* , if

$$\phi_n(x_1) = x_2, \dots, \phi_n(x_i) = x_{i+1}, \dots, \phi_n(x_k) = x_1.$$

Set

$$X_\sigma := \bigsqcup_{i=1}^k X_i \text{ where } X_i := \{x_i + \pi^n t; t \in C\} \subset \mathcal{O}_K/\pi^{n+1} \mathcal{O}_K.$$

Then

$$\phi_{n+1}(X_i) \subset X_{i+1} \ (1 \leq i \leq k-1) \text{ and } \phi_{n+1}(X_k) \subset X_1.$$

In the following we shall study the behavior of the finite dynamics  $\phi_{n+1}$  on the  $\phi_{n+1}$ -invariant set  $X_\sigma$  and determine all cycles in  $X_\sigma$  of  $\phi_{n+1}$ , which will be called *lifts* of  $\sigma$ . Remark that the length of any lift  $\tilde{\sigma}$  of  $\sigma$  is a multiple of  $k$ .

Let  $\mathbb{X}_i := x_i + \pi^n \mathcal{O}_K = \{x \in \mathcal{O}_K : x \equiv x_i \pmod{\pi^n}\}$  be the closed disk of radius  $p^{-n/e}$  centered at  $x_i \in \sigma$  and

$$\mathbb{X}_\sigma := \bigsqcup_{i=1}^k \mathbb{X}_i$$

be the clopen set corresponding to the cycle  $\sigma$ .

Let  $\psi := \phi^{\circ k}$  be the  $k$ -th iterate of  $\phi$ . Then any point in  $\sigma$  is fixed by  $\psi$ , the  $n$ -th induced map of  $\psi$ . For any point  $x \in \mathbb{X}_\sigma$ , we have  $(\psi(x) - x)/\pi^n \in \mathcal{O}_K$ . Using Taylor Expansion, for  $n \geq 1$ , we have

$$\psi(x + \pi^n t) \in x + \pi^n \left( \frac{\psi(x) - x}{\pi^n} \right) + \pi^n \psi'(x)t + \pi^{2n} \mathcal{O}_K, \quad \forall x \in \mathbb{X}_\sigma, \forall t \in \mathcal{O}_K.$$

Then it leads us to define the following two functions from  $\mathbb{X}_\sigma$  to  $\mathcal{O}_K$ . For  $x \in \mathbb{X}_\sigma$ , let

$$a_n(x) := \psi'(x) = \prod_{j=0}^{k-1} \phi'(\phi^{\circ j}(x)) \quad (4.2)$$

$$b_n(x) := \frac{\psi(x) - x}{\pi^n} = \frac{\phi^{\circ k}(x) - x}{\pi^n}. \quad (4.3)$$

The following lemma shows that the function  $a_n(x) \pmod{\pi^n}$  is always constant on  $\mathbb{X}_\sigma$ .

**Lemma 4.2.** *Let  $n \geq 1$  and  $\sigma = (x_1, \dots, x_k)$  be a  $k$ -cycle of  $\phi_n$ .*

(i) *For  $1 \leq i \leq k$ , we have*

$$a_n(x + t\pi^n) \equiv a_n(x) \pmod{\pi^n}, \quad \forall x \in \mathbb{X}_i, \forall t \in \mathcal{O}_K.$$

(ii) *For  $1 \leq i, j \leq k$ , we have*

$$a_n(x) \equiv a_n(y) \pmod{\pi^n}, \quad \forall x \in \mathbb{X}_i, \forall y \in \mathbb{X}_j.$$

(iii) For  $1 \leq i \leq k$ , we have

$$b_n(x) \equiv b_n(x + t\pi^n) \pmod{\pi^A}, \quad \forall x \in \mathbb{X}_i, \forall t \in \mathcal{O}_K.$$

where  $A := \min\{v_\pi(a_n(x) - 1), n\}$  for  $x \in \sigma$ .

(iv) If  $a_n(x) \not\equiv 0 \pmod{\pi}$  for some  $x \in \mathbb{X}_\sigma$ , then we have

$$\min\{v_\pi(b_n(x)), n\} = \min\{v_\pi(b_n(y)), n\}, \quad \forall x, y \in \mathbb{X}_\sigma.$$

Consequently,  $\min\{v_\pi(b_n(x)), A\} = \min\{v_\pi(b_n(y)), A\}$ .

*Proof.* Since  $\phi \in \mathcal{O}_K\langle x \rangle$ , it follows that  $\phi' \in \mathcal{O}_K\langle x \rangle$ . Then, the assertion (i) is a direct consequence of

$$a_n(x + t\pi^n) \equiv \prod_{j=0}^{k-1} \phi'(\phi^{\circ j}(x + t\pi^n)) \equiv \prod_{j=0}^{k-1} \phi'(\phi^{\circ j}(x)) \pmod{\pi^n},$$

where the second equality follows from that  $\phi' \in \mathcal{O}_K\langle x \rangle$ .

Assertion (ii) follows directly from the definition of  $a_n(x)$  and the facts that  $\phi' \in \mathcal{O}_K\langle x \rangle$  and  $\sigma = (x_i, \phi_n(x_i), \dots, \phi_n^{\circ k-1}(x_i))$ .

The 1-order Taylor Expansion of  $\psi$  at  $x$  gives

$$\psi(x + \pi^n t) - (x + \pi^n t) \equiv \pi^n \left( \frac{\psi(x) - x}{\pi^n} \right) + \pi^n t(\psi'(x) - 1) \pmod{\pi^{2n}}.$$

Hence

$$b_n(x + \pi^n t) \equiv b_n(x) + t(a_n(x) - 1) \pmod{\pi^n}. \quad (4.4)$$

Then (iii) follows.

Write

$$\psi(\phi(x)) - \phi(x) = \phi(\phi^{\circ k}(x)) - \phi(x) = \phi(x + \pi^n b_n(x)) - \phi(x).$$

The 1-order Taylor Expansion of  $\phi$  at  $x$  implies

$$\psi(\phi(x)) - \phi(x) \equiv \pi^n b_n(x) \phi'(x) \pmod{\pi^{2n}}.$$

Hence we have

$$b_n(\phi(x)) \equiv b_n(x) \phi'(x) \pmod{\pi^n}.$$

Thus we obtain (iv), because  $a_n(x) \not\equiv 0 \pmod{\pi}$  (for some  $x \in \mathbb{X}_\sigma$ ) implies  $\phi'(y) \not\equiv 0 \pmod{\pi}$  for all  $y \in \mathbb{X}_\sigma$ .  $\square$

The 1-order Taylor Expansion of  $\psi$  at  $x$  implies

$$\psi(x + \pi^n t) \equiv x + \pi^n b_n(x) + \pi^n a_n(x) t \pmod{\pi^{2n}}. \quad (4.5)$$

Define an affine map

$$\Psi : \mathbb{X}_\sigma \times (\mathcal{O}_K / \pi \mathcal{O}_K) \rightarrow \mathcal{O}_K / \pi \mathcal{O}_K$$

by

$$\Psi(x, t) = b_n(x) + a_n(x) t \pmod{\pi}.$$

An important consequence of formula (4.5) shows that  $\psi_{n+1} : X_i \rightarrow X_i$  is conjugate to the linear map

$$\Psi(x, \cdot) : \mathcal{O}_K / \pi \mathcal{O}_K \rightarrow \mathcal{O}_K / \pi \mathcal{O}_K$$

for some  $x \in \mathbb{X}_i$ .

We could call it the *linearization* of  $\psi_{n+1} : X_i \rightarrow X_i$ .

In order to determine the linearization of  $\psi_{n+1}$ , we only need to get the values modulo  $\pi$  of  $a_n(x)$  and  $b_n(x)$ . As we see in Lemma 4.2, the function  $a_n(x) \pmod{\pi}$  is always

constant on  $\mathbb{X}_\sigma$  and the function  $b_n(x) \pmod{\pi}$  is also constant on each  $\mathbb{X}_i$  but under the condition  $a_n(x) \equiv 1 \pmod{\pi}$ . For simplicity, sometimes we shall write  $a_n$  and  $b_n$  without mentioning  $x$  when we only care about the value of  $a_n(x) \pmod{\pi^n}$  and when we only care about  $b_n(x) \equiv 0 \pmod{\pi}$  or  $b_n(x) \not\equiv 0 \pmod{\pi}$  under  $a_n(x) \equiv 1 \pmod{\pi^n}$ .

We remind that the cardinality of the residual field  $\mathbb{K} = \mathcal{O}_K/\mathcal{P}_K = \mathcal{O}_K/\pi\mathcal{O}_K$  of  $K$  is  $p^f$ . The characteristic of the field  $\mathbb{K}$  is  $p$ . The multiplicative group  $\mathbb{K}^*$  is cyclic of order  $p^f - 1$ .

The above analysis allows us to distinguish the following four behaviors of  $\phi_{n+1}$  on  $X_\sigma$ :

(a) If  $a_n \equiv 1 \pmod{\pi}$  and  $b_n \not\equiv 0 \pmod{\pi}$ , then for any  $x \in \mathbb{X}_\sigma$ ,  $\Psi(x, \cdot)$  preserves  $p^{f-1}$  cycles of length  $p$ , so  $\phi_{n+1}$  restricted to  $X_\sigma$  preserves  $p^{f-1}$  cycles of length  $pk$ . In this case we say  $\sigma$  *grows*.

(b) If  $a_n \equiv 1 \pmod{\pi}$  and  $b_n \equiv 0 \pmod{\pi}$ , then for any  $x \in \mathbb{X}_\sigma$ ,  $\Psi(x, \cdot)$  is the identity, so  $\phi_{n+1}$  restricted to  $X_\sigma$  preserves  $p^f$  cycles of length  $k$ . In this case we say  $\sigma$  *splits*.

(c) If  $a_n \equiv 0 \pmod{\pi}$ , then for any  $x \in \mathbb{X}_\sigma$ ,  $\Psi(x, \cdot)$  is constant, so  $\phi_{n+1}$  restricted to  $X_\sigma$  preserves one cycle of length  $k$  and the remaining points of  $X_\sigma$  are mapped into this cycle. In this case we say  $\sigma$  *grows tails*.

(d) If  $a_n \not\equiv 0, 1 \pmod{\pi}$ , then for any  $x \in \mathbb{X}_\sigma$ ,  $\Psi(x, \cdot)$  is a permutation and the  $l$ -th iterate of  $\Psi(x, \cdot)$  reads

$$\Psi^{\circ l}(x, t) = b_n(a_n^l - 1)/(a_n - 1) + a_n^l t,$$

so that

$$\Psi^{\circ l}(x, t) - t = (a_n^l - 1) \left( t + \frac{b_n}{a_n - 1} \right).$$

Thus,  $\Psi(x, \cdot)$  admits a single fixed point  $t = -b_n/(a_n - 1)$ , and the remaining points lie on cycles of length  $\ell$ , where  $\ell$  is the order of  $a_n$  in  $(\mathcal{O}_K/\pi\mathcal{O}_K)^* = \mathbb{K}^*$ . So,  $\phi_{n+1}$  restricted to  $X_\sigma$  preserves one cycle of length  $k$  and  $\frac{p^f-1}{\ell}$  cycles of length  $k\ell$ . In this case we say  $\sigma$  *partially splits*.

Let  $\sigma = (x_1, \dots, x_k)$  be a  $k$ -cycle of  $\phi_n$  and let  $\tilde{\sigma}$  be a lift of  $\sigma$  of length  $kr$ , where  $r \geq 1$  is an integer. It follows immediately that  $\mathbb{X}_{\tilde{\sigma}} \subset \mathbb{X}_\sigma$ . For  $x \in \mathbb{X}_{\tilde{\sigma}}$ , we shall study the relation between  $(a_n, b_n)$  and  $(a_{n+1}, b_{n+1})$ . Our aim is to see the change of nature from a cycle to its lifts.

**Lemma 4.3.** *We have*

$$a_{n+1}(x) \equiv a_n^r(x) \pmod{\pi^n}, \quad (4.6)$$

$$\pi b_{n+1}(x) \equiv b_n(x)(1 + a_n(x) + \dots + a_n^{r-1}(x)) \pmod{\pi^n}. \quad (4.7)$$

*Proof.* The formula (4.6) follows from

$$a_{n+1}(x) \equiv (\psi^{\circ r})'(x) \equiv \prod_{j=0}^{r-1} \psi'(\psi^{\circ j}(x)) \equiv a_n^r(x) \pmod{\pi^n}.$$

By repeating  $r$ -times of the linearization (4.5), we obtain

$$\psi^{\circ r}(x) \equiv x + \Psi^{\circ r}(x, 0)\pi^n \pmod{\pi^{2n}},$$

where  $\Psi^{\circ r}(x, t)$  means the  $r$ -th composition of  $\Psi(x, r)$  as function of  $t$ . However,

$$\Psi^{\circ r}(x, 0) = b_n(x)(1 + a_n(x) + \dots + a_n^{r-1}(x)).$$

Thus (4.7) follows from the definition of  $b_{n+1}$  and the above two expressions.  $\square$

By Lemma 4.3, we immediately obtain the following proposition.

**Proposition 4.4.** *Let  $n \geq 1$ . Let  $\sigma$  be a  $k$ -cycle of  $\phi_n$  and  $\tilde{\sigma}$  be a lift of  $\sigma$ . Then we have*

- 1) *if  $a_n \equiv 1 \pmod{\pi}$ , then  $a_{n+1} \equiv 1 \pmod{\pi}$ ;*
- 2) *if  $a_n \equiv 0 \pmod{\pi}$ , then  $a_{n+1} \equiv 0 \pmod{\pi}$ ;*
- 3) *if  $a_n \not\equiv 0, 1 \pmod{\pi}$  and  $\tilde{\sigma}$  is of length  $k$ , then  $a_{n+1} \not\equiv 0, 1 \pmod{\pi}$ ;*
- 4) *if  $a_n \not\equiv 0, 1 \pmod{\pi}$  and  $\tilde{\sigma}$  is of length  $k\ell$  where  $\ell \geq 2$  is the order of  $a_n$  in  $(\mathcal{O}_K/\pi\mathcal{O}_K)^*$ , then  $a_{n+1} \equiv 1 \pmod{\pi}$ .*

This result is interpreted as follows in a dynamical way.

- i) If  $\sigma$  grows or splits, then any lift  $\tilde{\sigma}$  grows or splits.
- ii) If  $\sigma$  grows tails, then the single lift  $\tilde{\sigma}$  also grows tails.
- iii) If  $\sigma$  partially splits, then the lift  $\tilde{\sigma}$  of the same length as  $\sigma$  partially splits, and the other lifts of length  $k\ell$  grow or split.

By the above discussion, the case of growing tails is simple. If  $\sigma = (x_1, \dots, x_k)$  is a cycle of  $\phi_n$  which grows tails, then  $\phi$  admits a  $k$ -periodic point  $x_0$  in the clopen set  $\mathbb{X}_\sigma$  and  $\mathbb{X}_\sigma$  is contained in the attracting basin of the periodic orbit  $x_0, \phi(x_0), \dots, \phi^{o(k-1)}(x_0)$ .

Similarly, for the case of partially splitting, we can also find a periodic orbit in  $\mathbb{X}_\sigma$ , and other parts are reduced to the cases of growing and splitting. Hence, we mainly study these later cases.

For a cycle  $\sigma = (x_1, \dots, x_k)$  at level  $n$  and  $x \in \mathbb{X}_\sigma$ , let

$$A_n(x) := v_\pi(a_n(x) - 1), \quad B_n(x) := v_\pi(b_n(x)).$$

By Lemma 4.2,  $\hat{A}_n(x) := \min\{A_n(x), n\}$  does not depend on the choice of  $x \in \mathbb{X}_\sigma$  and if  $B_n(x) < \min\{A_n(x), n\}$  then  $B_n(x)$  does not depend on the choice of  $x \in \mathbb{X}_\sigma$ . Sometimes, there is no difference when we choose  $x \in \mathbb{X}_\sigma$ . So, without misunderstanding, we shall write  $A_n, \hat{A}_n$  and  $B_n$  without mentioning  $x$ .

First, we study the splitting case. We say a cycle *splits  $l$  times* if itself splits, its lifts split, the second generation of the descendants split, ..., and the  $(l-1)$ -th generation of the descendants split.

**Proposition 4.5.** *Let  $\sigma$  be a splitting cycle of  $\phi_n$ .*

- 1) *If  $\hat{A}_n > B_n$ , then every lift splits  $B_n - 1$  times then all lifts at level  $n + B_n$  grow.*
- 2) *If  $A_n \leq B_n$  and  $A_n < n$ , then there is one lift which behaves the same as  $\sigma$  (i.e., this lift splits and  $A_{n+1} \leq B_{n+1}$ ,  $A_{n+1} < n + 1$ ) and other lifts split  $A_n - 1$  times then all lifts at level  $n + A_n$  grow.*
- 3) *If  $B_n \geq n$  and  $A_n \geq n$ , then all lifts split at least  $n - 1$  times.*

*Proof.* Taking  $r = 1$  in Lemma 4.3, we have

$$a_{n+1}(x) - 1 \equiv a_n(x) - 1 \pmod{\pi^n} \quad (4.8)$$

and

$$\pi b_{n+1}(x) \equiv b_n(x) \pmod{\pi^n}. \quad (4.9)$$

- 1) Since  $\hat{A}_n > B_n$ , the assertion (ii) of Lemma 4.2 says that

$$b_n(x + \pi^n t) \equiv b_n(x) \pmod{\pi^{\hat{A}_n}}$$

for all  $t \in \mathcal{O}_K$ . Replacing  $x$  with  $x + \pi^n t$  in the equalities (4.8) and (4.9), we have  $B_{n+1}(x) = B_n(x) - 1$  and  $\hat{A}_{n+1}(x) \geq \hat{A}_n(x) \geq B_{n+1}(x)$  for all  $x \in \mathbb{X}_\sigma$ . So the lifts of  $\sigma$  split. By induction, the lifts will split  $B_n - 1$  times and  $B_{n+B_n} = 0$ . This means that  $b_{n+B_n}(x) \not\equiv 0 \pmod{\pi}$  for all  $x \in \mathbb{X}_\sigma$ . So all the descendants of  $\sigma$  at level  $n + B_n$  grow.



2) Since  $A_n \leq B_n$  and  $A_n < n$ , the equality (4.4) implies that there exists a lift  $\hat{\sigma}$  such that  $B_n(x) > B_n(y) = A_n$  for all  $x \in \mathbb{X}_{\hat{\sigma}}$  and  $y \in \mathbb{X}_{\sigma} \setminus \mathbb{X}_{\hat{\sigma}}$ . By equalities (4.8) and (4.9), we have  $B_{n+1}(x) \geq A_{n+1}(x)$  for all  $x \in \mathbb{X}_{\hat{\sigma}}$  and  $B_{n+1}(y) = A_n(y) - 1 < A_{n+1}(y)$  for all  $y \in \mathbb{X}_{\sigma} \setminus \mathbb{X}_{\hat{\sigma}}$ . So the lift  $\hat{\sigma}$  behaves the same as  $\sigma$ . By the proved assertion 1), the other lifts split  $A_n - 1$  times then all lifts at level  $n + A_n$  grow.

3) Since  $B_n \geq n$  and  $A_n \geq n$ , the equality (4.4) implies that

$$b_n(x) \equiv 0 \pmod{\pi^n}$$

for all  $x \in \mathbb{X}_{\sigma}$ . By the equality (4.9),  $B_{n+1}(x) \geq n - 1$ , so all the lifts split. By induction, all the lifts split at least  $n - 1$  times.  $\square$

Now we study the growing case. We begin with a lemma and a proposition.

**Lemma 4.6.** *Let  $\sigma$  be a growing cycle and let  $\tilde{\sigma}$  be a lift of  $\sigma$ . If  $n > v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x))$  for some  $x \in \mathbb{X}_{\sigma}$ , then  $n > v_{\pi}(1 + a_n(y) + \dots + a_n^{p-1}(y))$  for all  $y \in \mathbb{X}_{\sigma}$ .*

*Proof.* If  $A_n(x) \geq n$ , then  $a_n(x) \equiv 1 \pmod{\pi^n}$ . The condition  $n > v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x))$  implies  $n > e$ . According to Lemma 4.3, we have  $a_n(y) \equiv 1 \pmod{\pi^n}$  for all  $y \in \mathbb{X}_{\sigma}$ . Thus  $n > v_{\pi}(1 + a_n(y) + \dots + a_n^{p-1}(y)) = e$ .

If  $A_n(x) < n$ , then the condition  $n > v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x))$  implies that  $v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x)) = v_{\pi}(1 + a_n(y) + \dots + a_n^{p-1}(y))$  for all  $y \in \mathbb{X}_{\sigma}$ . So  $n > v_{\pi}(1 + a_n(y) + \dots + a_n^{p-1}(y))$ .  $\square$

**Proposition 4.7.** *Let  $\sigma$  be a growing cycle and  $\tilde{\sigma}$  be a lift of  $\sigma$ . Then if  $n > v_{\pi}(1 + a_n + \dots + a_n^{p-1})$ , we have  $\tilde{\sigma}$  splits  $v_{\pi}(1 + a_n + \dots + a_n^{p-1}) - 1$  times then all the descendants grow.*

*Proof.* Taking  $r = p$  in Lemma 4.3, we have

$$\begin{aligned} a_{n+1}(x) - 1 &\equiv a_n(x)^p - 1 \pmod{\pi^n} \\ &\equiv (a_n(x) - 1)(1 + a_n(x) + \dots + a_n^{p-1}(x)) \pmod{\pi^n}. \end{aligned}$$

So we have  $A_{n+1}(x) \geq \min\{n, A_n(x) + v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x))\}$ .

Since  $\sigma$  grows,  $b_n(x) \not\equiv 0 \pmod{\pi}$ . By Lemma 4.3, we have

$$\pi b_{n+1}(x) \equiv b_n(x)(1 + a_n(x) + \dots + a_n^{p-1}(x)) \pmod{\pi^n}. \quad (4.10)$$

Hence if  $n > v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x))$ ,

$$B_{n+1}(x) = v_{\pi}(b_{n+1}(x)) = v_{\pi}(1 + a_n(x) + \dots + a_n^{p-1}(x)) - 1.$$

Therefore, by the assertion 1) of Proposition 4.5,  $\tilde{\sigma}$  splits  $v_{\pi}(1 + a_n + \dots + a_n^{p-1}) - 1$  times then all the descendants grow.  $\square$

Lemma 4.6 and Proposition 4.7 lead to the following proposition.

**Proposition 4.8.** *Suppose  $n \geq e + 1$ . If  $\sigma$  grows and  $A_n(x) > e/(p - 1)$  for some  $x \in \sigma$ , then all the  $p^{f-1}$  lifts of  $\sigma$  split  $e - 1$  times and then all the descendants at level  $n + e$  grow.*

*Proof.* Since  $\sigma$  grows and  $A_n(x) > e/(p - 1)$ , we can distinguish two cases :  $a_n \equiv 1 \pmod{\pi^n}$  and  $a_n = 1 + \pi^{\gamma}\delta$  with  $|\delta|_p = 1$  and  $e/(p - 1) < \gamma < n$ .

If  $a_n \equiv 1 \pmod{\pi^n}$ , then by  $n \geq e + 1$ , we have  $1 + a_n + \dots + a_n^{p-1} \equiv p \pmod{\pi^n}$ .

If  $a_n = 1 + \pi^\gamma \delta$  with  $|\delta|_p = 1$  and  $e/(p-1) < \gamma < n$ , then we have

$$\begin{aligned} 1 + a_n + \cdots + a_n^{p-1} &= \frac{a_n^p - 1}{a_n - 1} = \frac{\binom{p}{1}\pi^\gamma \delta + \binom{p}{2}\pi^{2\gamma} \delta^2 + \cdots + \pi^{\gamma p} \delta^p}{\pi^\gamma \delta} \\ &= p + \binom{p}{2}\pi^\gamma \delta + \cdots + \pi^{\gamma(p-1)} \delta^{p-1} \\ &\equiv p \pmod{\pi^{e+\gamma}}. \end{aligned}$$

Thus both cases imply  $v_\pi(1 + a_n + \cdots + a_n^{p-1}) = e$ . Since  $n > e$ , by Lemma 4.6 and Proposition 4.7, all the  $p^{f-1}$  lifts of  $\sigma$  split  $e-1$  times then all the descendants at level  $n+e$  grow.  $\square$

Now we give a technique lemma which will be useful later.

**Lemma 4.9.** *Let  $\sigma$  be a growing  $k$ -cycle at level  $n \geq e+1$  and set  $\gamma = \hat{A}_n$ .*

- 1) *If  $\gamma > e/(p-1)$ , then  $v_\pi(1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j}) = e$ , for all  $j \geq 0$ .*
- 2) *If  $\gamma \leq e/(p-1)$ , then*

$$v_\pi(1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j}) = \begin{cases} p^j \gamma (p-1) & \text{for } 0 \leq j < \log_p \frac{e}{\gamma(p-1)}; \\ e & \text{for } j \geq \log_p \frac{e}{\gamma(p-1)}. \end{cases}$$

*Proof.* 1) By the definition of  $\gamma$ , we have  $\gamma \leq n$ .

If  $\gamma = n$ , then  $a_n \equiv 1 \pmod{\pi^n}$  which implies  $a_n^{p^j} \equiv 1 \pmod{\pi^n}$ . Thus we have  $1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j} \equiv p \pmod{\pi^n}$ . Then by  $n \geq e+1$ , the assertion follows.

If  $e/(p-1) < \gamma < n$ , then by putting  $a_n = 1 + \pi^\gamma \delta_0$  with  $|\delta_0|_p = 1$ , we have

$$a_n^{p^j} = 1 + \sum_{i=1}^{p^j} \binom{p^j}{i} (\pi^\gamma \delta_0)^i. \quad (4.11)$$

Since  $\gamma > e/(p-1)$ , the equation (4.11) implies

$$v_\pi(a_n^{p^j} - 1) > e/(p-1).$$

Let  $\gamma_j = v_\pi(a_n^{p^j} - 1)$  and write  $a_n^{p^j} = 1 + \pi^{\gamma_j} \delta_j$  with  $|\delta_j|_p = 1$ . Then we have

$$1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j} = \frac{a_n^{p^{j+1}} - 1}{a_n^{p^j} - 1} \quad (4.12)$$

$$= \frac{\binom{p}{1}\pi^{\gamma_j} \delta_j + \binom{p}{2}\pi^{2\gamma_j} \delta_j^2 + \cdots + \pi^{p\gamma_j} \delta_j^p}{\pi^{\gamma_j} \delta_j} \quad (4.13)$$

$$= p + \binom{p}{2}\pi^{\gamma_j} \delta_j + \cdots + \pi^{(p-1)\gamma_j} \delta_j^{p-1} \quad (4.14)$$

$$\equiv p \pmod{\pi^{e+1}}. \quad (4.15)$$

Thus,

$$v_\pi(1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j}) = e.$$

- 2) For  $0 \leq j < \log_p(e/((p-1)\gamma))$ , the formula (4.11) implies

$$a_n^{p^j} \equiv 1 + (\pi^\gamma \delta_0)^{p^j} \pmod{\pi^{e+1}}.$$

So  $v_\pi(a_n^{p^j} - 1) = p^j \gamma < e/(p-1)$ .

Let  $\gamma_j = v_\pi(a_n^{p^j} - 1)$  and write  $a_n^{p^j} = 1 + \pi^{\gamma_j} \delta_j$  with  $|\delta_j|_p = 1$ . By (4.13) and (4.14), we have

$$\begin{aligned} 1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j} &= \frac{a_n^{p^{j+1}} - 1}{a_n^{p^j} - 1} \\ &\equiv p + \pi^{(p-1)\gamma_j} \delta_j^{(p-1)} \pmod{\pi^{e+1}}. \end{aligned}$$

Thus,

$$v_\pi(1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j}) = \gamma_j(p-1) < e.$$

For  $j > \log_2(e/\gamma)$ , by (4.13) and (4.14), we have

$$\begin{aligned} 1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j} &= \frac{a_n^{p^{j+1}} - 1}{a_n^{p^j} - 1} \\ &\equiv p \pmod{\pi^{e+1}}. \end{aligned}$$

So,

$$v_\pi(1 + a_n^{p^j} + \cdots + a_n^{(p-1)p^j}) = e.$$

□

Lemma 4.9 gives the following proposition on the behavior of a growing cycle and all of its descendants.

**Proposition 4.10.** *Let  $\sigma$  be a growing cycle at level  $n$ . If  $n \geq e + 1$  and  $v_\pi(a_n - 1) > e/(p-1)$ , then any lift of  $\sigma$  splits  $e-1$  times then all the descendants grow; and again the lifts of the growing cycles split  $e-1$  times then the descendants grow. This process will go on forever.*

Keep the assumption and notation of Proposition 4.10. If  $K$  is a non-trivial extension of  $\mathbb{Q}_p$ , i.e.,  $K \neq \mathbb{Q}_p$ , then the clopen set  $\mathbb{X}_\sigma$  will be decomposed into infinitely many, more precisely, uncountably many minimal components.

In fact, if  $K \neq \mathbb{Q}_p$ , we have  $e > 1$ , or  $f > 1$ . In the case  $e > 1$ , there are always  $e-1$  times of splitting during the consecutive growings. Each splitting at some level decomposes each set at this level into  $p^f$  parts invariant under  $\phi$ . Since we have infinitely many times of splitting, we will have uncountably many minimal parts.

In the case of  $f > 1$ , since the lift of a growing cycle has length multiplied by  $p$  only, the growing cycle becomes  $p^{f-1}$  cycles each time. Thus each set at the growing level is decomposed into  $p^{f-1}$  invariant components of  $\phi$ . By the same reason, we will also have uncountably many minimal parts.

Moreover, we can see each component is not a finite union of balls any more. They are Cantor type sets. The clopen set  $\mathbb{X}_\sigma$  with these growing and splitting behaviors is called of *type*  $(k, e)$ .

In general, for a given sequence of positive numbers  $\vec{E} = (E_j)_{j \geq 0}$ , the clopen set  $\mathbb{X}_\sigma$  is called of *type*  $(k, \vec{E})$  if it is a growing  $k$ -cycle at level  $n$  and the lifts of this  $k$ -cycle split  $E_0 - 1$  times. Furthermore, all the  $E_0$ -th generation of descendants grow and then all the lifts split  $E_1 - 1$  times. Again, the descendants grow and the lifts of the growing descendants split  $E_2 - 1$  times, ....

Let  $(p_s)_{s \geq 1}$  be a sequence of positive integers with the property that  $p_s | p_{s+1}$  for all  $s \geq 0$ . Consider the inverse limit

$$\mathbb{Z}_{(p_s)} := \varprojlim \mathbb{Z}/p_s \mathbb{Z}.$$

This is a profinite group, usually called an *odometer*. The map  $\tau : x \mapsto x + 1$  is called the *adding machine* on  $\mathbb{Z}_{(p_s)}$ .

It can be shown that if  $\mathbb{X}_\sigma$  is of type  $(k, \vec{E})$ , then  $(\mathbb{X}_\sigma, \phi)$  can be decomposed into uncountably many minimal subsystems. The power series  $\phi$  restricted to each minimal component is the adding machine with the sequence  $(p_s)$  equal to

$$(p_s) = (k, \underbrace{kp, \dots, kp}_{E_0}, \underbrace{kp^2, \dots, kp^2}_{E_1}, \underbrace{kp^3, \dots, kp^3}_{E_2}, \dots).$$

We remark that the type  $(k, e)$  is nothing but the type  $(k, \vec{E})$  with  $\vec{E} = (e, e, \dots)$ .

As a consequence of Lemma 4.9, we have the following proposition.

**Proposition 4.11.** *Let  $\sigma$  be a growing  $k$ -cycle at level  $n \geq e + 1$  and set  $\gamma = \hat{A}_n$ .*

- 1) *If  $\gamma > e/(p-1)$ , then the clopen set  $\mathbb{X}_\sigma$  is of type  $(k, e)$ .*
- 2) *If  $\gamma \leq e/(p-1)$  and  $\log_p \frac{e}{\gamma(p-1)}$  is not an integer, then the clopen set  $\mathbb{X}_\sigma$  is of type  $(k, \vec{E})$ , with*

$$E_j = \begin{cases} p^j \gamma (p-1), & \text{if } 0 \leq j < \log_p \frac{e}{\gamma(p-1)}; \\ e, & \text{if } j > \log_p \frac{e}{\gamma(p-1)}. \end{cases}$$

- 3) *If  $\gamma \leq e/(p-1)$  and  $\log_p \frac{e}{\gamma(p-1)}$  is an integer, then all the lifts of  $\sigma$  split  $\gamma(p-1) - 1$  times then all the descendants grow; and again the lifts of the growing cycles split  $\gamma(p-1)p^j - 1$  times then the descendants grow; ..., the lifts of the growing cycles (at level  $n - \gamma + e/(p-1)$ ) split at least  $e - 1$  times.*

## 5. MINIMAL DECOMPOSITION

In this section, we show how we can do minimal decomposition for a convergent series  $\phi \in \mathcal{O}_K\langle x \rangle$ .

If a cycle grows tails, it will produce an attracting periodic orbit with an attracting basin. If a splitting cycle always has a splitting lift with the same length, then it will produce a periodic orbit of  $\phi \in \mathcal{O}_K\langle x \rangle$ . For a growing cycle  $\sigma$  at level  $n \geq e + 1$ , if  $\hat{A}_n > e/(p-1)$  and  $K \neq \mathbb{Q}_p$ , then it will produce uncountably many minimal components of  $\phi$ .

We shall describe these assertions more precisely.

Let  $\sigma = (x_1, \dots, x_k)$  be a cycle of  $\phi_n$ . Recall that in this case  $\sigma$  is called a  $k$ -cycle at level  $n$ . There are four special situations for the dynamical system  $\phi : \mathbb{X}_\sigma \rightarrow \mathbb{X}_\sigma$ .

(S1) Suppose  $\sigma$  grows tails. Then  $\phi$  admits a  $k$ -periodic orbit with one periodic point in each ball  $\mathbb{X}_i = \{x \in \mathcal{O}_K : x \equiv x_i \pmod{\pi^n}\} (1 \leq i \leq k)$ , and all other points in  $\mathbb{X}_\sigma$  are attracted into this orbit. In this situation, if  $x$  is a point in the  $k$ -periodic orbit, then  $|(\phi^{\circ k})'(x)|_p < 1$  since  $(\phi^{\circ k})'(x) = a_m(x) \equiv 0 \pmod{\pi}$  for all  $m \geq n$ . The periodic orbit  $(x, \phi(x), \dots, \phi^{\circ(k-1)}(x))$  is then attractive.

(S2) Suppose  $\sigma$  grows at level  $n \geq e + 1$  and  $\hat{A}_n > e/(p-1)$ . By Proposition 4.10, any lift of the growing cycle  $\sigma$ , splits  $e - 1$  times then all the descendants grow; and again the lifts of the growing cycles split  $e - 1$  times then the descendants grow. This process will go on forever. The dynamical system  $(\mathbb{X}_\sigma, \phi)$  which consists of uncountably many minimal subsystem, is of type  $(k, e)$ .

(S3) Suppose  $\sigma$  splits and there is a splitting lift with the same length as  $\sigma$  at level  $n+1$ , and then for this lift there is still a splitting lift with the same length, and go on. Then there is a  $k$ -periodic orbit with one periodic point in each  $\mathbb{X}_i$  ( $1 \leq i \leq k$ ). We say that  $\sigma$  is a starting splitting cycle at level  $n$ . In this situation, if  $x$  is a point in the  $k$ -periodic orbit, then  $|(\phi^{\circ k})'(x)|_p = 1$  since  $(\phi^{\circ k})'(x) = a_m(x) \equiv 1 \pmod{\pi}$  for all  $m \geq n$ . Thus the periodic orbit  $(x, \phi(x), \dots, \phi^{\circ(k-1)}(x))$  is indifferent.

(S4) Suppose  $\sigma$  partially splits. Then by Lemma 4.3, there is one lift of length  $k$  which still partially splits like  $\sigma$ . Thus there is a  $k$ -periodic orbit with one periodic point in each  $\mathbb{X}_i$  ( $1 \leq i \leq k$ ). In this situation, if  $x$  is a point in the  $k$ -periodic orbit formed above, then  $|(\phi^{\circ k})'(x)|_p = 1$  since  $(\phi^{\circ k})'(x) = a_m(x) \not\equiv 0, 1 \pmod{\pi}$  for all  $m \geq n$ . Hence, the periodic orbit  $(x, \phi(x), \dots, \phi^{\circ(k-1)}(x))$  is indifferent.

Before proving Theorem 1.1, we first show that there are only finitely many possible periods of periodic orbits. Let  $\sigma = (x_1, \dots, x_k)$  be a cycle at level  $n$ . Recall that  $\hat{A}_n(x) = \min\{A_n(x), n\}$  does not depend on the choice of  $x \in \mathbb{X}_\sigma$ . So, without misunderstanding, we will not mention  $x$  in  $\hat{A}_n(x)$ .

**Proposition 5.1.** *Let  $\sigma = (x_1, \dots, x_k)$  be a growing cycle at level  $n \geq e+1$ .*

- 1) *If  $\log_p \frac{e}{(p-1)\hat{A}_n}$  is not a nonnegative integer, then there is no periodic point in  $\mathbb{X}_\sigma$ .*
- 2) *If  $\log_p \frac{e}{(p-1)\hat{A}_n}$  is a nonnegative integer, then the possible periods of the periodic orbits in  $\mathbb{X}_\sigma$  must be  $\frac{kpe}{(p-1)\hat{A}_n}$ .*

*Proof.* 1) The conclusion is a direct consequence of the assertions 1) and 2) of Proposition 4.11.

2) Let  $\sigma_s$  be a cycle which is a descendant of  $\sigma$  at level  $s := n - \hat{A}_n + ep/(p-1)$ . By the third assertion of Proposition 4.11,  $\sigma_s$  is a growing or splitting cycle of length  $\frac{kpe}{(p-1)\hat{A}_n}$  with  $\hat{A}_s \geq \min\{ep/(p-1), n\} > e$ .

If the cycle  $\sigma_s$  grows, then there is no periodic point in the clopen set  $\mathbb{X}_{\sigma_s} = \bigsqcup_{x \in \sigma} (x + \pi^s \mathcal{O}_K)$ .

If the cycle  $\sigma_s$  splits, we first suppose that there is a splitting lift with the same length as  $\sigma$  at level  $n+1$ , and then for this lift there is still a splitting lift with the same length, and go on. Then there is a  $\frac{kpe}{(p-1)\hat{A}_n}$ -periodic orbit with one periodic point in each  $x + \pi^s \mathcal{O}_K$  ( $x \in \sigma_s$ ). If it is not the case, let  $\sigma_t$  be a growing descendant of  $\sigma_s$  at level  $t > s$  with the same length as  $\sigma_s$ . Since  $\hat{A}_s > e$ , it follows  $\hat{A}_t \geq e$ . So the clopen set  $\mathbb{X}_{\sigma_t} = \bigsqcup_{x \in \sigma_t} (x + \pi^t \mathcal{O}_K)$  is of type  $(\frac{kpe}{(p-1)\hat{A}_n}, e)$ . Thus there is no periodic point in  $\mathbb{X}_{\sigma_t}$ . □

**Proposition 5.2.** *Let  $\sigma = (x_1, \dots, x_k)$  be a splitting cycle at level  $n \geq e+1$ . Then the possible periods of periodic orbits in  $\mathbb{X}_\sigma$  must be  $k$  or  $\frac{kpe}{(p-1)\hat{A}_n}$  (when  $\log_p \frac{e}{(p-1)\hat{A}_n}$  is a nonnegative integer).*

*Proof.* Suppose first that there is a splitting lift with the same length as  $\sigma$  at level  $n+1$ , and then for this lift there is still a splitting lift with the same length, and go on. Then there is a  $k$ -periodic orbit with one periodic point in each  $x_i + \pi^n \mathcal{O}_K$  ( $x_i \in \sigma$ ).

If not, let  $\sigma_s$  be a growing descendant of  $\sigma$  at level  $s > n$  with the same length to  $\sigma$ . As in Proposition 5.1, we have two cases.

- 1) If  $\log_p \frac{e}{(p-1)\hat{A}_n}$  is not a nonnegative integer, then neither is  $\log_p \frac{e}{(p-1)\hat{A}_s}$ . So there is no periodic point in the clopen set  $\mathbb{X}_{\sigma_s} = \bigsqcup_{x \in \sigma_s} (x + \pi^s \mathcal{O}_K)$ .
- 2) If  $\log_p \frac{e}{(p-1)\hat{A}_n}$  is a nonnegative integer, then so is  $\log_p \frac{e}{(p-1)\hat{A}_s}$ . By Proposition 5.1, the possible periods of periodic orbits in  $\mathbb{X}_{\sigma_s}$  must be  $\frac{kpe}{(p-1)\hat{A}_n}$ .  $\square$

An immediate consequence of Propositions 5.1 and 5.2 is the following corollary.

**Corollary 5.3.** *For each  $\phi \in \mathcal{O}_K \langle x \rangle$ , there are only finitely many possible periods of periodic orbits in  $\mathcal{O}_K$ .*

Now we can prove our main theorem.

*Proof of Theorem 1.1.* Let us first prove that there are only finitely many periodic points. In fact, there are only finitely many possible lengths of periods by Corollary 5.3. But periodic points are solutions of the equations  $\phi^{\circ q_i}(x) = x$  with  $\{q_i\}$  being one of the finite possible lengths of periods. Since  $\mathcal{O}_K \langle x \rangle$  is closed under composition of functions, it follows  $\phi^{\circ q_i} \in \mathcal{O}_K \langle x \rangle$ . Write

$$\phi^{\circ q_i}(x) - x = \sum_{i \geq 0} c_i x^i$$

and let  $j$  be the largest integer such that  $|c_j|_p = \max_{i \geq 0} |c_i|_p$ . By Proposition 3.2, there is a monic polynomial  $g \in \mathcal{O}_K[x]$  of degree  $j$  and a power series  $h \in \mathcal{O}_K \langle x \rangle$  such that  $\phi^{\circ q_i}(x) - x = g(x)h(x)$ , and  $h(x) \neq 0$  for all  $x \in \mathcal{O}_K$ . Since  $g$  is a polynomial, the equation  $g(x) = 0$  admits only a finite number of solutions. Hence, there are only a finite number of periodic points.

Let us continue the proof. We start from the level  $e + 1$ . The space  $\mathcal{O}_K$  is a union of  $p^{(e+1)f}$  balls with radius  $p^{-(e+1)/e}$ . Each ball is identified with a point in  $\mathcal{O}_K / \pi^{e+1} \mathcal{O}_K$ . The induced map  $\phi_{e+1}$  on  $\mathcal{O}_K / \pi^{e+1} \mathcal{O}_K$  admits some cycles. The points outside the cycles are mapped into the cycles. The balls corresponding to the points outside the cycles will be put into the third part  $C$  in the decomposition. From now on, we will be concerned only with cycles at level  $n \geq e + 1$ .

Let  $\sigma = (x_1, \dots, x_k)$  be a cycle at level  $n \geq e + 1$ . We distinguish four cases.

(P1)  $\sigma$  grows tails. Then by (S1), the clopen set  $\mathbb{X}_\sigma$  consists of a  $k$ -periodic orbit and other points are attracted by this periodic orbit. So,  $\mathbb{X}_\sigma$  contributes to the first part  $A$  and the third part  $C$  in the decomposition.

(P2)  $\sigma$  grows. We shall apply Lemma 4.9 and Proposition 4.11.

- (1) If  $\hat{A}_n > e/(p-1)$ , then the clopen set  $\mathbb{X}_\sigma$  is of type  $(k, e)$ . So,  $\mathbb{X}_\sigma \subset B$  consists some minimal components of the part  $B$  in the decomposition.
- (2) If  $\hat{A}_n \leq e/(p-1)$  and  $\log_p \frac{p}{(p-1)\hat{A}_n}$  is not an integer, then the clopen set  $\mathbb{X}_\sigma = \bigsqcup_{i=1}^k (x_i + \pi^n \mathcal{O}_K)$  is of type  $(k, \vec{E})$ , with

$$E_j = \begin{cases} p^j(p-1)\hat{A}_n, & \text{if } 0 \leq j < \log_p \frac{e}{(p-1)\hat{A}_n}; \\ e, & \text{if } j > \log_p \frac{e}{(p-1)\hat{A}_n}. \end{cases}$$

Consider the level  $s := n + \hat{A}_n p^{\lfloor \log_p \frac{e}{(p-1)\hat{A}_n} \rfloor + 1} - \hat{A}_n$ . Let  $\sigma_s$  be any cycle at level  $s$  which is a descendant of  $\sigma$  at level  $n$ . Then  $\sigma_s$  is a growing cycle of length  $\ell := kp^{\lfloor \log_p \frac{e}{(p-1)\hat{A}_n} \rfloor + 1}$ . The clopen set  $\mathbb{X}_{\sigma_s} = \bigsqcup_{x \in \sigma_s} (x + \pi^s \mathcal{O}_K)$  is then of type  $(\ell, e)$ . So  $\mathbb{X}_{\sigma_s} \subset B$  and  $\mathbb{X}_\sigma \subset B$ .

- (3) If  $\hat{A}_n \leq e/(p-1)$  and  $\log_p \frac{p}{(p-1)\hat{A}_n}$  is an integer, then we need to study the descendants of  $\sigma$  at level  $s = n - \hat{A}_n + ep/(p-1)$  separately. Let  $\sigma_s$  be such a descendant. Then  $\sigma_s$  is a growing or splitting cycle of length  $\ell := \frac{kpe}{(p-1)\hat{A}_n}$  with  $\hat{A}_s \geq \min\{ep/(p-1), n\} > e$ . If  $\sigma_s$  grows, then we go to the case of (1). If  $\sigma_s$  splits, then we go to (P3).

(P3)  $\sigma$  splits. We shall apply Proposition 4.5.

- (1) If  $\sigma$  belongs to Case 1 of Proposition 4.5, then after finitely many times of splitting, all lifts grow. So all the lifts are in some case of (P2) determined by  $\hat{A}_n$ .
- (2) If  $\sigma$  belongs to Case 2 of Proposition 4.5, then there is one lift of  $\sigma$  sharing the property (S3), and there is a periodic orbit with period  $k$ . Each lift different from the cycle deriving the periodic orbit (at any level  $m \geq n+1$ ) find itself in some case of (P2) (determined by  $\hat{A}_n$ ) after finitely many times of lifting.
- (3) If  $\sigma$  belongs to Case 3 of Proposition 4.5, then  $\sigma$  splits into  $p^{n_f}$  cycles at level  $2n$ . These cycles at level  $2n$  may continue this procedure of analysis of (P3). But this procedure can not continue infinitely, because there are only a finite number of periodic points. So, all these cycles may continue to split but they must end with their lifts belonging either to Case 1 or Case 2 in Proposition 4.5. So,  $\mathbb{X}_\sigma$  contributes to both  $A$  and  $B$ .

(P4)  $\sigma$  partially splits. In this case,  $\sigma$  is in the situation (S4). Thus there comes out a periodic orbit. For  $m \geq n$ , we suppose that  $\sigma_{m+1}$  is the lift of  $\sigma$  at level  $m+1$  deriving the periodic orbit. Then the other lifts different from  $\sigma_{m+1}$  will grow or split. So we go to (P2) or (P3) for these cycles at level  $m+1$ .

For the case (3) of (P2), a splitting lift  $\sigma_s$  of  $\sigma$  at level  $s = n - \hat{A}_n + ep/(p-1)$  must have  $\hat{A}_s > e$ . By the process of (P3), the growing descendants of  $\sigma_s$  are in the case (1) of (P2), so the procedures will stop. Thus, all the above procedures will stop and we get the decomposition.  $\square$

## 6. AFFINE POLYNOMIAL DYNAMICS

In this section, we would like to study the minimal decomposition of the affine polynomial dynamical system  $(K, F)$  with  $F(x) = \alpha x + \beta$ ,  $(\alpha, \beta \in K, \alpha \neq 0, (\alpha, \beta) \neq (1, 0))$  by using the idea of cycle lifting.

For  $F(x) = \beta$  or  $F(x) = x$ , the dynamical system  $(K, F)$  is trivial.

We distinguish two cases:  $F$  is a translation or not.

**Case I.** If  $F(x) = x + \beta$  with  $\beta \neq 0$ , then  $F$  is conjugate to the translation  $\hat{F}(x) = x + 1$  by  $h(x) = x/\beta$ .

We consider  $\mathcal{O}_K$  as a single point cycle at level 0, denoted by  $(0)$ . We say that the cycle  $(0)$  at level 0 grows if  $\mathcal{O}_K/\pi\mathcal{O}_K$  consists of  $p^{f-1}$  cycles of length  $p$  under  $F_1$ . Then  $(\mathcal{O}_K, F)$  is said to be of type  $(1, e)$  at level 0 if the cycle  $(0)$  grows at level 0, the  $p^{f-1}$  lifts of  $(0)$  with length  $p$  at level 1 splits  $e-1$  times then all the descendants grow; and again the lifts of the growing cycles split  $e-1$  times then the descendants grow; and so on.

For  $a \in K$ , denote by  $\mathbb{B}(a, 1) = \{x \in K, |x - a|_p \leq 1\}$  the ball centered at  $a$  with radius 1. We have the following theorem.

**Theorem 6.1.** *For the translation  $F(x) = x + 1$  acting on  $K$ , each ball of radius 1 is  $F$ -invariant. For  $a \in K$ , each subsystem  $(\mathbb{B}(a, 1), F)$  is conjugate to  $(\mathcal{O}_K, F)$  by  $h(x) = x - a$ . The dynamical system  $(\mathcal{O}_K, F)$  is of type  $(1, e)$  at level 0.*

**Case II.** If  $F(x) = \alpha x + \beta$ ,  $\alpha \neq 1$ , then  $F$  is conjugate to the multiplication  $\hat{F}(x) = \alpha x$  by  $h(x) = x - (\beta/(1 - \alpha))$ . Thus we need only to study the multiplication dynamics  $(K, F(x) = \alpha x)$ . It is easy to see that  $0, \infty$  are two fixed points of  $F$ .

Let  $\mathbb{U} := \{x \in \mathcal{O}_K : |x|_p = 1\}$  be the group of units in  $\mathcal{O}_K$  and  $\mathbb{V} := \{x \in \mathbb{U} : \exists m \in \mathbb{N}, m \geq 1, x^m = 1\}$  be the set of roots of unity in  $\mathcal{O}_K$ . The set  $\mathbb{V}$  is a finite set of  $(p^f - 1)p^s$  elements, with  $p^s$  being the highest power of  $p$  such that  $K$  has a root of unity of order  $p^s$ . For the details, see the book of Robert [20]. We will distinguish three cases:

$$(A) \alpha \notin \mathbb{U}, \quad (B) \alpha \in \mathbb{V} \setminus \{1\}, \quad \text{and} \quad (C) \alpha \in \mathbb{U} \setminus \mathbb{V}.$$

The following we will treat the three cases separately. The first two cases are easy.

Case (A)  $\alpha \notin \mathbb{U}$ . If  $|\alpha|_p < 1$ , then the system admits an attracting fixed point  $0$  with the whole  $K$  being the attracting basin, that is,

$$\lim_{n \rightarrow \infty} F^{\circ n}(x) = 0, \quad \forall x \in K.$$

If  $|\alpha|_p > 1$ , then the system admits a repelling fixed point  $0$  and the whole  $K$  except  $0$  lies in the attracting basin of  $\infty$ , that is,

$$\lim_{n \rightarrow \infty} F^{\circ n}(x) = \infty, \quad \forall x \in K \setminus \{0\}.$$

Case (B)  $\alpha \in \mathbb{V} \setminus \{1\}$ . Let  $\ell$  be the order of  $\alpha$ , i.e., the least integer such that  $\alpha^\ell = 1$ . It is easy to see that all points are in a periodic orbit with period  $\ell$ .

Case (C)  $\alpha \in \mathbb{U} \setminus \mathbb{V}$ . For each  $n \in \mathbb{Z}$ , the sphere  $\pi^{-n}\mathbb{U}$  is  $F$ -invariant. The dynamical system  $(\pi^{-n}\mathbb{U}, F)$  is conjugate to the system  $(\mathbb{U}, F)$  by  $h(x) = \pi^n x$ .

$$\begin{array}{ccc} \pi^{-n}\mathbb{U} & \xrightarrow{\alpha x} & \pi^{-n}\mathbb{U} \\ \pi^n x \downarrow & & \downarrow \pi^n x \\ \mathbb{U} & \xrightarrow{\alpha x} & \mathbb{U} \end{array}$$

Now we are going to study the dynamical system  $(\mathbb{U}, F)$ .

**Theorem 6.2.** Consider the system  $(\mathbb{U}, F)$  where  $F(x) = \alpha x$  with  $\alpha \in \mathbb{U} \setminus \mathbb{V}$ . Let  $\ell$  be the order of  $\alpha$  in  $\mathbb{K}^*$ . One can decompose  $\mathbb{U}$  into  $(p^f - 1)p^{v_\pi(\alpha^\ell - 1) \cdot f - f} / \ell$  clopen sets such that each clopen set is of type  $(\ell, \vec{E})$  at level  $v_\pi(\alpha^\ell - 1)$  where

$$\vec{E} = \left( v_\pi\left(\frac{\alpha^{\ell p} - 1}{\alpha^\ell - 1}\right), v_\pi\left(\frac{\alpha^{\ell p^2} - 1}{\alpha^{\ell p} - 1}\right), \dots, v_\pi\left(\frac{\alpha^{\ell p^{N+1}} - 1}{\alpha^{\ell p^N} - 1}\right), e, e, \dots \right).$$

Here  $N$  is the largest integer such that  $v_\pi((\alpha^{\ell p^{N+1}} - 1)/(\alpha^{\ell p^N} - 1)) \neq e$ .

*Proof.* Notice that  $\ell$  is the order of  $\alpha$  in  $\mathbb{K}^*$ . Thus  $\alpha^\ell \equiv 1 \pmod{\pi}$ . We can check that there are  $(p^f - 1)/\ell$  cycles of length  $\ell$  at level 1. Now we consider the  $\ell$ -cycles at level 1.

For each  $x_0 \in \mathbb{U}$ ,

$$\begin{aligned} a_1(x_0) &= (F^{\circ \ell})'(x_0) = \alpha^\ell, \\ b_1(x_0) &= \frac{F^{\circ \ell}(x_0) - x_0}{\pi} = \frac{(\alpha^\ell - 1)x_0}{\pi}. \end{aligned}$$

Thus  $a_1(x_0) \equiv 1 \pmod{\pi}$  and  $v_\pi(b_1(x_0)) = v_\pi(\alpha^\ell - 1) - 1$ . By induction, one can check that each  $\ell$ -cycle at level 1 splits  $v_\pi(\alpha^\ell - 1) - 1$  times and then all its lifts grow.



Let  $m = v_\pi(\alpha^\ell - 1) + 1$ . Consider the  $\ell p$ -cycles at level  $m$ . For each  $x_0 \in \mathbb{U}$ ,

$$\begin{aligned} a_m(x_0) &= (F^{\circ \ell p})'(x_0) = \alpha^{\ell p}, \\ b_m(x_0) &= \frac{F^{\circ \ell p}(x_0) - x_0}{\pi^m} = \frac{(\alpha^{\ell p} - 1)x_0}{\pi^m}. \end{aligned}$$

Then we have

$$v_\pi(b_m(x_0)) = v_\pi(\alpha^{\ell p} - 1) - m = v_\pi\left(\frac{\alpha^{\ell p} - 1}{\alpha^\ell - 1}\right) - 1.$$

Hence each  $\ell p$ -cycle splits  $v_\pi\left(\frac{\alpha^{\ell p} - 1}{\alpha^\ell - 1}\right) - 1$  times and then all its lifts grow.

Now let  $q = v_\pi(\alpha^{\ell p} - 1) + 1$ . Consider the  $\ell p^2$ -cycles at level  $q$ . By the same calculations, we have for each  $x_0$ ,

$$a_q(x_0) = \alpha^{\ell p^2} \equiv 1 \pmod{\pi}$$

and

$$v_\pi(b_q(x_0)) = v_\pi(\alpha^{\ell p^2} - 1) - q = v_\pi\left(\frac{\alpha^{\ell p^2} - 1}{\alpha^{\ell p} - 1}\right) - 1.$$

Hence each  $\ell p^2$ -cycle splits  $v_\pi\left(\frac{\alpha^{\ell p^2} - 1}{\alpha^{\ell p} - 1}\right) - 1$  times and then all its lifts grow.

Go on this process, we can show that each  $\ell p^k$ -cycle at level  $v_\pi(\alpha^{\ell p^{k-1}} - 1) + 1$  splits  $v_\pi\left(\frac{\alpha^{\ell p^k} - 1}{\alpha^{\ell p^{k-1}} - 1}\right) - 1$  times and then all its lifts grow.

Since  $v_\pi(\alpha^{\ell p^k} - 1) \rightarrow \infty$  when  $k \rightarrow \infty$  for  $1 \leq i \leq p-1$  (see [21], p.100), and  $v_\pi(p) = e$ , we can find an integer  $N$  such that for all  $k > N$ , we have

$$\begin{aligned} v_\pi\left(\frac{\alpha^{\ell p^{k+1}} - 1}{\alpha^{\ell p^k} - 1}\right) - 1 &= v_\pi(1 + \alpha^{\ell p^k} + \cdots + \alpha^{(p-1)\ell p^k}) - 1 \\ &= v_\pi((1 - 1) + (\alpha^{\ell p^k} - 1) + \cdots + (\alpha^{(p-1)\ell p^k} - 1) + p) - 1 \\ &= v_\pi(p) - 1 = e - 1. \end{aligned}$$

Therefore, we can conclude that each  $\ell$ -cycle at level  $v_\pi(\alpha^\ell - 1)$  is of type  $(\ell, \vec{E})$  with

$$\vec{E} = \left( v_\pi\left(\frac{\alpha^{\ell p} - 1}{\alpha^\ell - 1}\right), v_\pi\left(\frac{\alpha^{\ell p^2} - 1}{\alpha^{\ell p} - 1}\right), \dots, v_\pi\left(\frac{\alpha^{\ell p^{N+1}} - 1}{\alpha^{\ell p^N} - 1}\right), e, e, \dots \right).$$

On the other hand, we have that the number of  $\ell$ -cycle at level  $v_\pi(\alpha^\ell - 1)$  is

$$\frac{p^f - 1}{\ell} \cdot (p^f)^{v_\pi(\alpha^\ell - 1) - 1},$$

since one  $\ell$ -cycle splits into  $p^f$  number of  $\ell$ -cycles. This completes the proof.  $\square$

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